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The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills

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Abstract: We derive the one loop mixing matrix for anomalous dimensions in $\mathcal{N} = 4$ Super Yang-Mills. We show that this matrix can be identified with the hamiltonian of an integrable SO(6) spin chain with vector sites. We then use the Bethe ansatz to find a recipe for computing anomalous dimensions for a wide range of operators. We give exact results for BMN operators with two impurities and results up to and including first order $1/J$ corrections for BMN operators with many impurities. We then use a result of Reshetikhin's to find the exact one-loop anomalous dimension for an SO(6) singlet in the limit of large bare dimension. We also show that this last anomalous dimension is proportional to the square root of the string level in the weak coupling limit.

Keywords: 1/N Expansion, Penrose limit and pp-wave background, Bethe Ansatz.

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Contents

1. Introduction

One of the main results of the AdS/CFT correspondence is that individual string states are mapped to local gauge-invariant operators in a dual field theory [1]–[3]. But even in the most well understood case of $\mathcal{N} = 4$ Super Yang-Mills (SYM) this mapping is only known for a small subset of the operators. The difficulty in making this mapping explicit is two-fold: i) String quantization on an $AdS_5 \times S^5$background is still unsolved. ii) The spectrum of gauge invariant operators is somewhat difficult to compute.

Previously, it was known that the chiral primaries in the gauge theory are dual to the string states that survive the supergravity limit. More recently it was realized how to go beyond the chiral primaries by considering operators with large R-charges, J [4]. On the string side this corresponds to semiclassical states with large angular momentum on the S^5. For such states, the $AdS_5 \times S^5$geometry essentially reduces via a Penrose limit to a plane wave geometry [5]–[7]. String theory on the plane wave background is solvable [8, 9] and an identification can be made between the string states and the gauge invariant operators. The string quantization on the plane wave is simple enough, at least in the light-cone gauge, where all string states are generated by an infinite set of creation operators similar to those in flat space [8, 9].

Amazingly, the operators dual to each of the eigenstates of the light-cone string hamiltonian can be identified. These (BMN) operators are [4]:

$$
\begin{aligned}
|0;J\rangle &\Longleftrightarrow \operatorname{tr} Z^J\,,\\
a_0^{i\,\dagger}\,|0;J\rangle &\Longleftrightarrow \operatorname{tr}\Phi_i Z^J,\\
a_n^{i\,\dagger} a_{-n}^{j\,\dagger}\,|0;J\rangle &\Longleftrightarrow \sum_l \mathrm{e}^{\,2\pi i l n/J}\operatorname{tr}\Phi_i Z^l \Phi_j Z^{J-l}\,,
\end{aligned}
$$

and so on. Here, Φ_i, $i = 1, \ldots, 6$ are the six scalar fields of $\mathcal{N} = 4$SYM in the adjoint representation of SU(N), and $Z = \Phi_1 + i\Phi_2$. The BMN operators have charge J under the generator of the R symmetry group, which rotates Φ_1 into Φ_2. On the string side, J is essentially the length of the string on the light-cone. The chain of Zs can be regarded as a field-theory realization of the string, which emerges as a compound of J constituents, much in the spirit of the string-bit models [10]. String excitations are represented by impurities inserted in the chain [4].

String theory makes a prediction for the anomalous dimensions of the BMN operators at any value of the Yang-Mills coupling in the large-N limit, by equating the mass of a string state with the full dimension of an operator [4]. This prediction can be verified by explicit perturbative calculations [4, 11, 12]. Furthermore, one can incorporate stringy corrections in the effective string coupling J^2/N and compare the results of the string calculations with the gauge theory computations [13]–[35].

Inverting the logic we can say that by resolving the mixing of operators with two or more impurities, order by order in perturbation theory, one can reconstruct the string spectrum by computing the anomalous dimensions of operators. We will follow this logic in an attempt to better understand the operator/string correpondence for a wider class of string states, including those states that are outside of the semiclassical regime [36]. These states would correspond to operators made of scalar fields and with high engineering dimension but in low representations of SO(6).

In this paper we will consider mixing of generic scalar operators $\operatorname{tr}\Phi_{i_1}\ldots\Phi_{i_L}$ to one-loop order in SYM perturbation theory. The problem appears difficult, not only because the number of operators grows rapidly with L (roughly as 6^L), but also because the operators mix in a way which at first sight seems hopelessly entangled. However, we are able to make progress in solving this problem by establishing an equivalence of the mixing matrix with the hamiltonian of a certain integrable spin chain. This equivalence will allow us to use powerful techniques of the algebraic Bethe ansatz [37]–[40] to diagonalize the mixing matrix. In particular, we will find that the problem of finding the one-loop anomalous dimensions comes down to solving a set of Bethe equations.

Among the results contained in this paper, we are able to reproduce easily recent results [41] for the one loop anomalous dimensions of BMN operators with two impurities. We then extend these results to a large class of BMN operators with more than two impurities. We are able to identify BMN states with the corresponding Bethe states, where among other things, we show that a "bound state" containing M Bethe roots extending into the complex plane corresponds to having string states with M identical oscillators. We also give a recipe for finding $1/J$ corrections to the anomalous dimensions including the explicit results for the first order corrections. These corrections are important since they correspond to curvature corrections away from the plane-wave background in the full $AdS_5 \times S^5$ [42]–[44].

We then go beyond the BMN limit in two explicit examples. The first example corresponds to an SO(6) singlet made up of L scalar fields. In the large L limit this can be solved explicitly [45], and in fact corresponds to the operator made up only of scalars that has the largest anomalous dimension for bare dimension L. We find the anomalous dimension and demonstrate that it is linear in L. We also argue that the string level behaves roughly as L^2, so the full dimension of the operator is proportional to the square-root of the level, a result that follows from AdS string theory in strong coupling for generic operators [2]. The second example is the direct analog of the Heisenberg anti-ferromagnet, where we also find the anomalous dimension and show that it is linear in L. We also show how to put in "holes" on these states and explicitly compute the changes in the anomalous dimensions coming from the holes. The holes can be either SO(6) vectors or one of the SO(6) spinors.

Integrable structures have previously appeared in string theory for generalizations of the plane-wave background [46]–[48]. It is not clear if there is a relation between this integrability and the integrability discussed in this paper. But it might indicate that the integrability encountered here is not accidental but is a manifestation of some general principle yet to be found. We should also mention that integrable spin chains arise in perturbative analysis of Regge scattering in large-N QCD and Bethe ansatz techniques were extensively applied there [49]–[53].

In section 1 we derive the one loop mixing matrix for all scalar operators. In section 2 we use this matrix to compute the anomalous dimensions for a few simple examples. In section 3 we give a brief review of Reshetikhin's proof of integrability for the SO(6) vector chain and his solution for the eigenvalues of the transfer matrix in terms of Bethe roots, along with the Bethe equations the roots must satisfy. In section 4 we use the results from the previous section to compute the anomalous dimensions for two impurities to all orders in $1/J$ and for many impurities to first order in $1/J$. In section 5 we describe solutions to the Bethe equations [45] which correspond to operators outside the BMN limit. We compute the anomalous dimensions for these operators and for nearby operators. In section 6 we give our conclusions.

2. Anomalous dimensions from the spin system

We will study one-loop renormalization for all scalar operators without derivatives:

$$\mathcal{O}[\psi] = \psi^{i_1 \dots i_L} \operatorname{tr} \Phi_{i_1} \dots \Phi_{i_L} \,. \tag{2.1}$$

Many interesting operators in N=4 SYM, notably chiral primary and BMN operators, belong to this class. In general, the scalar operators (2.1) mix under renormalization. There is a distinguished basis, in which operators are multiplicatively renormalizable. It is important that up to possible degeneracies, rotations to this basis will diagonalize the two-point correlation functions. As far as one-loop renormalization is concerned, the scalar operators will mix only among themselves. Mixing with other operators should occur at higher orders in perturbation theory.

Renormalized operators in general are linear combinations of bare operators. If we choose the particular operator basis,

$$\mathcal{O}^A_{\rm ren} = Z^A{}_B \mathcal{O}^B \,, \tag{2.2}$$

then we can find the renormalization factor by requiring finitness of the correlation function

$$\left\langle Z_\Phi^{1/2} \Phi_{j_1}(x_1) \cdots Z_\Phi^{1/2} \Phi_{j_L}(x_L) \; \mathcal{O}^A_{\rm ren}(x) \right\rangle \,. \tag{2.3}$$

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Here, Z_Φ is the wave-function renormalization factor, that is multiplication by Z_Φ makes the two-point correlator $\langle \Phi_i \Phi_j \rangle$ finite. All renormalization factors depend on the UV cutoff Λ and on the 't Hooft coupling in the large-N limit. By standard arguments, the renormalization factor determines the matrix of anomalous dimensions through

$$\Gamma = \frac{dZ}{d \ln \Lambda} \cdot Z^{-1} . \tag{2.4}$$

Eigenvectors of Γ correspond to operators which are multiplicatively renormalizable. The corresponding eigenvalues determine the anomalous dimensions of these operators. Thus,

$$\langle \mathcal{O}_n(x) \mathcal{O}_n(y) \rangle = \frac{\text{const.}}{|x-y|^{2(L+\gamma_n)}} \tag{2.5}$$

for the operator that corresponds to an eigenvector of Γ with an eigenvalue γ_n.

How should one characterize the Hilbert space[1] of scalar operators of bare dimension L? Let us forget for a moment the cyclicity of the trace. Then in the natural basis (2.1) each operator is associated with an SO(6) tensor with L indices. Such tensors form a 6^L-dimensional linear space $\mathcal{H} = V_1 \otimes \ldots \otimes V_L$, where $V_l = \mathbb{R}^6$ is associated with an SO(6) index in the lth position in $\psi^{i_1 \ldots i_l \ldots i_L}$. The anomalous dimensions are thus eigenvalues of a $6^L \times 6^L$ matrix. It will prove extremely useful to regard $\mathcal{H}$ as a Hilbert space of a *spin system*. That is, let us consider a one-dimensional lattice with L sites whose ends are identified and let each lattice site host a six-dimensional real vector. The space of states for such a spin system is isomorphic to $\mathcal{H}$. The matrix of anomalous dimensions is a hermitean operator in $\mathcal{H}$ and can be regarded as a hamiltonian of the spin system. Recalling that wave functions which differ by a cyclic permutation of indices correspond to the same operator, we should impose the constraint that physical states have zero total momentum:

$$U \left| \psi \right\rangle = \left| \psi \right\rangle , \tag{2.6}$$

where U is the translation operator

$$U \, a_1 \otimes \cdots \otimes a_{L-1} \otimes a_L = a_L \otimes a_1 \otimes \cdots \otimes a_{L-1} . \tag{2.7}$$

In the strict large-N limit, all operators (2.1) are independent and there are no other constraints.

With the spin system interpretation in mind, let us compute the matrix of anomalous dimensions at one loop. The renormalization of BMN operators with two impurities was extensively discussed, so the essential pieces of the calculation for the anomalous dimensions are present throughout the literature (e.g. [4, 13, 11, 15]). We will therefore skip many details and give only salient features of the derivation, generalizing to arbitrary scalar operators. We use the standard Feynman rules which follow from the euclidean SYM action:

$$S = \frac{1}{g^2} \int d^4x \, \operatorname{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i)^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 + \text{fermions} \right\} , \tag{2.8}$$

and we will work in the Feynman gauge, in which the scalar and the gauge boson propagators are equal, up to Lorentz and SO(6) structures.

[1] We shall call it a Hilbert space, even though it is finite-dimensional.

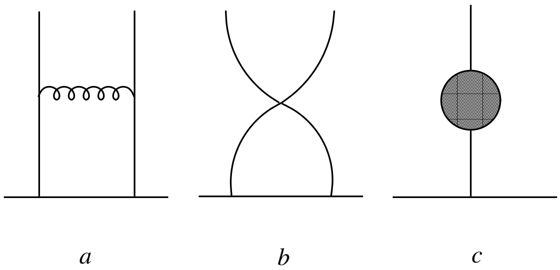
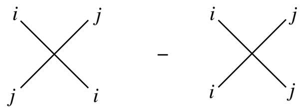


Figure 1: One-loop diagrams.

There are three types of planar one-loop diagrams that contribute to the correlation function (2.3) (figure 1). We depict the operator $\mathcal{O}[\psi]$ by a horizontal bar with scalar propagators ending on each of the scalar fields (i.e. lattice sites) in the operator (2.1). Only lattice sites affected by loop corrections are shown in the figure. Since the gauge boson exchange is flavor-blind, the Z factor associated with diagram (a) is diagonal in SO(6) indices:

$$Z^{(a)\dots j_l j_{l+1}\dots}_{\dots i_l i_{l+1}\dots} = I - \frac{\lambda}{16\pi^2} \ln\Lambda \, \delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} \, .$$

The SO(6) structure of the Z factor arising from diagram (b) can be easily inferred from the structure of the quartic scalar vertex:



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Thus we find that

$$Z^{(b)\dots j_l j_{l+1}\dots}_{\dots i_l i_{l+1}\dots} = I - \frac{\lambda}{16\pi^2} \ln\Lambda \left(2\delta^{j_{l+1}}_{i_l} \delta^{j_l}_{i_{l+1}} - \delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} - \delta_{i_l i_{l+1}} \delta^{j_l j_{l+1}} \right) \, .$$

The one-loop self-energy correction in diagram (c) leads to the wave-function renormalization. The corresponding renormalization factor was computed in Feynman gauge [54] and is given by

$$Z_\Phi = 1 + \frac{\lambda}{4\pi^2} \ln\Lambda \, .$$

One half of the self-energy corrections in the correlation function (2.3) are cancelled by wave-function renormalization of the external legs. The remaining divergence should be cancelled by renormalization of the operator. The corresponding Z factor is proportional to the unit matrix, and can be written as

$$Z^{(c)\dots j_l j_{l+1}\dots}_{\dots i_l i_{l+1}\dots} = I + \frac{\lambda}{8\pi^2} \ln\Lambda \, \delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} \, .$$

Adding all the pieces together, we find that the contribution from each link of the lattice is

$$Z^{\dots j_l j_{l+1}\dots}_{\dots i_l i_{l+1}\dots} = I + \frac{\lambda}{16\pi^2} \ln\Lambda \left(\delta_{i_l i_{l+1}} \delta^{j_l j_{l+1}} + 2\delta^{j_l}_{i_l} \delta^{j_{l+1}}_{i_{l+1}} - 2\delta^{j_{l+1}}_{i_l} \delta^{j_l}_{i_{l+1}} \right) \, . \qquad (2.9)$$

The total Z factor is the product over all links of the expression in (2.9).

The matrix of anomalous dimensions can be expressed in terms of two elemtary operators which act on each link: the trace operator,

$$K_{i_l i_{l+1}}^{j_l j_{l+1}} = \delta_{i_l i_{l+1}} \delta^{j_l j_{l+1}} \,, \tag{2.10}$$

and the permutation operator:

$$P_{i_l i_{l+1}}^{j_l j_{l+1}} = \delta_{i_l}^{j_{l+1}} \delta_{i_{l+1}}^{j_l} \,. \tag{2.11}$$

These operators act in the tensor product $\mathbb{R}^6 \otimes \mathbb{R}^6$ as

$$\begin{aligned} K\, a \otimes b &= (a \cdot b) \sum_i \widehat{\mathrm{e}}^i \otimes \widehat{\mathrm{e}}^i \,, \\ P\, a \otimes b &= b \otimes a \,, \end{aligned} \tag{2.12}$$

where $\widehat{\mathrm{e}}^i$ are a set of orthogonal unit vectors in $\mathbb{R}^6$. The matrix of anomalous dimensions is

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(K_{l,l+1} + 2 - 2P_{l,l+1} \right) \,, \tag{2.13}$$

where the subscripts indicate that the operators act in the tensor product of nearest-neighbor spins $V_l \otimes V_{l+1}$. By introducing the spin operators

$$M_{ij}^{ab} = \delta_i^a \delta_j^b - \delta_j^a \delta_i^b \tag{2.14}$$

for each lattice site, we can rewrite the hamiltonian in the form in which spin-spin interactions are manifest:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left[M_l^{ab} M_{l+1}^{ab} - \frac{1}{16} \left(M_l^{ab} M_{l+1}^{ab} \right)^2 + \frac{9}{4} \right] . \tag{2.15}$$

The result in (2.13) for the matrix of anomalous dimensions in the form of a hamiltonian of a spin system is the main result of this section.

Examples

The hamiltonian in (2.13) posesses some remarkable properties. We will see in the next section that it belongs to a unique series of integrable spin chains with SO(n) symmetry. For an arbitrary SO(n) spin chain, integrability requires that the ratio of coefficients between the permutation operator and the trace operator is $-(n/2-1)$. For SO(6), this ratio is -2, precisely matching the ratio in (2.13)! Integrability allows one to use powerful techniques of the Bethe ansatz to diagonalize the hamiltonian and compute its eigenvalues. The review of the Bethe ansatz for the SO(6) spin chain is given in the next section.

Since the Bethe ansatz utilizes rather sophisticated algebraic constructions, we would first like to demonstrate the formalism by rederiving known results for some of the simpler operators before invoking the Bethe ansatz machinery.

The simplest and most important scalar operators in N=4 SYM are chiral primaries, operators which are symmetric and traceless in all SO(6) indices. Chiral primaries are annihilated by the trace operator K in (2.10) and are eigenstates of the permutation operator with an eigenvalue one. Therefore,

$$\Gamma\,|CPO\rangle = 0\,, \tag{2.16}$$

which reflects the fact that scaling dimensions of chiral primaries are protected by supersymmetry and should not receive quantum corrections.

Another interesting operator is the Konishi scalar,

$$KO = \operatorname{tr}\Phi_i\Phi_i\,. \tag{2.17}$$

It is also invariant under permutations, but now the trace operator acts non-trivially: $K\,|KO\rangle = 6\,|KO\rangle$. The Konishi operator corresponds to the lattice with two sites. Each link between the lattice sites gives an equal contribution to the anomalous dimension, so

$$\Gamma\,|KO\rangle = \frac{3\lambda}{4\pi^2}\,|KO\rangle\ , \tag{2.18}$$

in agreement with the calculation of [55].

Consider now BMN operators with two impurities:

$$\mathcal{O}_{ij} = \sum_{l=0}^{J}\psi_l\ \operatorname{tr}\Phi_i Z^l \Phi_j Z^{J-l} \qquad (i \neq j\,,\quad i,j = 3,\ldots,6)\,. \tag{2.19}$$

The spin-chain hamiltonian acts on such operators as a lattice Schrödinger operator with δ' potential:

$$(\Gamma\psi)_l = -\frac{\lambda}{4\pi^2}\left[\psi_{l+1} + \psi_{l-1} - 2\psi_l + \frac{1}{2}\left(\delta_{l0} - \delta_{lJ}\right)\left(\psi_0 - \psi_J\right)\right]. \tag{2.20}$$

The exact (multiplicatively renormalizable at any J) BMN operators with two impurities were recently found by Biesert [41]. His operators correspond to taking

$$\psi_l^S = \cos\left[\frac{(2l+1)n\pi}{J+1}\right] \tag{2.21}$$

for states which are symmetric under interchange of i and j, and

$$\psi_l^A = \sin\left[\frac{2(l+1)\pi n}{J+2}\right] \tag{2.22}$$

for antisymmetric states. It is straightforward to check that the above states are eigenfunctions of Γ with eigenvalues

$$\gamma_n^S = \frac{\lambda}{\pi^2}\sin^2\left(\frac{\pi n}{J+1}\right) \tag{2.23}$$

and

$$\gamma_n^A = \frac{\lambda}{\pi^2}\sin^2\left(\frac{\pi n}{J+2}\right)\,. \tag{2.24}$$

As explained in [41], symmetric and antisymmetric operators with the same J belong to different supermultiplets and for that reason their anomalous dimensions are different.

Finally, there are singlet BMN operators of the form:

$$\mathcal{O} = \sum_{l=0}^{J} \phi_l \sum_{i=3}^{6} \operatorname{tr} \Phi_i Z^l \Phi_i Z^{J-l} - \chi \operatorname{tr} \bar{Z} Z^{J+1} \,. \tag{2.25}$$

The matrix of anomalous dimensions acts on these operators as

$$\begin{aligned} (\Gamma\phi)_l &= -\frac{\lambda}{4\pi^2}\left[\phi_{l+1} + \phi_{l-1} - 2\phi_l - \frac{1}{2}\left(\delta_{l0} + \delta_{lJ}\right)\left(\phi_0 + \phi_J - \chi\right)\right] \\ \Gamma\chi &= -\frac{\lambda}{4\pi^2}(\phi_0 + \phi_J - \chi)\,. \end{aligned} \tag{2.26}$$

This is a Schrödinger operator with a self-consistent source and a repulsive δ-function potential. Note that the source and the potential come from the trace term in the spin-chain hamiltonian. Operators constructed in [41] correspond to the wave functions

$$\begin{aligned} \phi_l &= \cos\left[\frac{(2l+3)\pi n}{J+3}\right], \\ \chi &= 2\cos\left(\frac{\pi n}{J+3}\right). \end{aligned} \tag{2.27}$$

It is easy to check that they are eigenfunctions of the hamiltonian with eigenvalues

$$\gamma_n = \frac{\lambda}{\pi^2}\sin^2\left(\frac{\pi n}{J+3}\right), \tag{2.28}$$

in agreement with the anomalous dimensions computed in [41].

3. A short review of the Bethe ansatz equations

In this section we review the Yang-Baxter equation, the construction of commuting operators and the Bethe-ansatz for an SO(n) chain where all sites in the chain transform in the vector representation.[2]

In order to find an integrable system, one needs to construct an R-matrix. An R-matrix $R_{12}(u)$ acts on a tensor product of two n dimensional vector spaces, $V_1 \otimes V_2$. The parameter u is the *spectral parameter* and the matrix elements are explicitly given by $R_{12}(u)^{i_1 i_2}_{j_1 j_2}$. The *transfer matrix* $T(u)$ is constructed from the R-matrix as

$$T(u) = R_{01}(u)R_{02}(u)R_{03}(u)\cdots R_{0L}(u)\,. \tag{3.1}$$

Here, the transfer matrix acts on the tensor product of $L+1$ n-dimensional vector spaces. The sites on the chain are numbered from 1 to L while the space V_0 is an auxilary space.One can think of $T(u)$ as a matrix of operators that act on the L sites of the chain, with the different matrix elements given by ${T^{i_0}}_{j_0}(u)$.

[2]For a nice explanation of the Yang-Baxter equation and the algebraic Bethe ansatz see [56].

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If a system is integrable, then the R-matrix satisfies the *Yang-Baxter equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)\,, \tag{3.2}$$

where the three R-matrices act on the tensor product of three n-dimensional vector spaces. Given the Yang-Baxter equation, one can find the corresponding relation for a product of transfer matrices

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v)\,, \tag{3.3}$$

where the indices a and b refer to two different auxillary spaces, but the transfer matrices act on the same chain of L sites. Writing the components of the auxillary spaces explicitly, (3.3) becomes

$$T_a^{i_a}{}_{j_a}(u)T_b^{i_b}{}_{j_b}(v) = R_{ab}^{-1\,i_a i_b}{}_{k_a k_b}(u-v)T_b^{k_b}{}_{l_b}(v)T_a^{k_a}{}_{l_a}(u)R_{ab}^{l_a l_b}{}_{j_a j_b}(u-v)\,. \tag{3.4}$$

Taking the trace on the $V_a \otimes V_b$ tensor space, we get

$$\mathrm{Tr}_a(T_a(u))\,\mathrm{Tr}_b(T_b(v)) = \mathrm{Tr}_b(T_b(v))\,\mathrm{Tr}_a(T_a(u))\,, \tag{3.5}$$

and since the auxillary spaces are traced over, we can drop the labels a and b and write

$$[\mathrm{Tr}(T(u)), \mathrm{Tr}(T(v))] = 0 \tag{3.6}$$

for all u and v. For the case that we will be considering it will turn out that these traces are order $2L$ polynomials in the spectral parameter, hence the Yang-Baxter equation implies that there are up to $2L$ independent operators that are mutually commuting.

Consider then the R-matrix acting on $V_1 \otimes V_2$

$$R_{12} = \frac{1}{n-2}[u(2u+2-n)I_{12} - (2u+2-n)P_{12} + 2uK_{12}]\,, \tag{3.7}$$

where I_{12}, P_{12} and K_{12} are the identity, exchange and trace operators defined in the previous section. This R-matrix satisfies the Yang-Baxter equation [39, 40]. The verification for this is straightforward, but tedious.

Clearly, the transfer matrices will be polynomials of order $2L$ in u, which we write as

$$T(u) = \sum_m u^m T_m\,. \tag{3.8}$$

The traces we write as

$$t(u) \equiv \mathrm{Tr}(T(u)) = \sum_m u^m t_m\,. \tag{3.9}$$

Let us find the first few terms in the expansion. Since $R_{12}(0) = P_{12}$, the lowest order term in the expansion is

$$T_0 = \prod_{\ell=1}^{L} P_{0\ell}\,. \tag{3.10}$$

Hence the action of this operator on the tensor product of the $L+1$ vector spaces is

$$T_0 V_0 \otimes V_1 \otimes \cdots \otimes V_{L-1} \otimes V_L = V_1 \otimes V_2 \otimes \cdots \otimes V_L \otimes V_0\,. \tag{3.11}$$

If we now take the trace over the V_0 space, we have that

$$t_0 V_1 \otimes \cdots \otimes V_{L-1} \otimes V_L = V_2 \otimes \cdots \otimes V_L \otimes V_1 \,. \tag{3.12}$$

Hence t_0 is the discrete shift operator, the operator that shifts everything by one site. We already encountered this operator in imposing the cyclicity of the trace on the SYM operators.

The next term in (3.8) is found by replacing one $P_{0\ell}$ operator in (3.10) with

$$-I_{0\ell} - \frac{2}{n-2} P_{0\ell} + \frac{2}{n-2} K_{0\ell} \tag{3.13}$$

and summing over all positions ℓ.The contribution from $P_{0\ell}$ will just give us the shift operator again. To find the contributions of the other operators, note that

$$\mathrm{Tr}_0 \left(\prod_{k=1}^{\ell-1} P_{0k} \prod_{k=\ell+1}^{L} P_{0k} \right) = t_0 P_{\ell,\ell+1} \tag{3.14}$$

and

$$\mathrm{Tr}_0 \left(\prod_{k=1}^{\ell-1} P_{0k} K_{0\ell} \prod_{k=\ell+1}^{L} P_{0k} \right) = t_0 K_{\ell,\ell+1} \,. \tag{3.15}$$

Hence we find that t_1 is given by

$$t_1 = \frac{2}{n-2} t_0 \left(\sum_{\ell=1}^{L} (K_{\ell,\ell+1} - 1 - \frac{n-2}{2} P_{\ell,\ell+1}) \right) . \tag{3.16}$$

Since t_1 and t_0 are among a set of commuting operators and since we are free to add a constant, we see that

$$\sum_{\ell=1}^{L} \left(K_{\ell,\ell+1} + \frac{n-2}{2} - \frac{n-2}{2} P_{\ell,\ell+1} \right) \tag{3.17}$$

also commutes with these operators.

If we now consider the particular case of SO(6), we see that (3.17) is proportional to the anomalous dimension operator in (2.4)! Therefore, the one-loop anomalous dimension operator described in the previous section can be mapped to a hamiltonian of an integrable system.

Showing that a hamiltonian is part of an integrable system is only part of the story. We also want to find the eigenstates and the eigenvalues of $t(u) = \mathrm{Tr}\, T(u)$. In the Heisenberg spin chain, this is done most efficiently by using the algebraic Bethe ansatz. One can use the algebraic Bethe ansatz for the SO(n) chain as well [57, 58].However, as was shown by Reshetikhin [39, 40], there is another way to find the eigenvalues of $t(u)$ which are constrained by a series of Bethe equations.

Let us give a brief sketch of Reshetikhin's argument. The first thing to observe is that the R-matrix in (3.7) has a crossing symmetry

$$(R_{12}(u))^{T_2} = R_{12} \left(-u + \frac{n-2}{2} \right) , \tag{3.18}$$

where T_2 signifies a transpose on V_2 only. Assuming that u is real, it is then straightforward to show that

$$(t(u))^{\dagger} = t\left(-u + \frac{n-2}{2}\right). \tag{3.19}$$

Hence, the eigenvalues $\Lambda(u)$ of $t(u)$ satisfy

$$\overline{\Lambda}(u) = \Lambda\left(-u + \frac{n-2}{2}\right). \tag{3.20}$$

Next consider the combination of R-matrices

$$R_{12}\left(\frac{n-2}{2}\right) R_{13}\left(u + \frac{n-2}{2}\right) R_{23}(u) = K_{12} R_{13}\left(u + \frac{n-2}{2}\right) R_{23}(u). \tag{3.21}$$

If we define $K_{12}^{\perp}$ as the orthogonal complement to the trace operator, then by the Yang-Baxter equation we have that

$$K_{12} R_{13}\left(u + \frac{(n-2)}{2}\right) R_{23}(u) K_{12}^{\perp} = 0. \tag{3.22}$$

This means that $R_{13}(u+\frac{n-2}{2})R_{23}(u)$ can be written in lower triangular form on the $V_1 \otimes V_2$ space, where the upper left block corresponds to the operator $K_{12}R_{13}(u+\frac{n-2}{2})R_{23}(u)K_{12}$ and the right lower block to $K_{12}^{\perp}R_{13}(u+\frac{n-2}{2})R_{23}(u)K_{12}^{\perp}$

We next note that

$$\begin{aligned} R_{13}\left(u + \frac{n-2}{2}\right)^{i_1 i}{}_{j_1 k} R_{23}(u)^{i_1 k}{}_{j_1 j} &= \\ = \frac{1}{(n-2)^2}\Big[(4u^2 - (n-2)^2)(A^{i_1 i}{}_{j_1 j} + u^2 B^{i_1 i}{}_{j_1 j}) + 4u^2 C^{i_1 i}{}_{j_1 j}\Big], \end{aligned} \tag{3.23}$$

where only the k index is summed over and where

$$\begin{aligned} A^{i_1 i}{}_{j_1 j} &= -\delta^{i_1 i}\delta^{i_1}{}_j \\ B^{i_1 i}{}_{j_1 j} &= \delta^{i_1}{}_{j_1}\delta^{i}{}_j \\ C^{i_1 i}{}_{j_1 j} &= n\delta^{i_1}{}_{j_1}\delta^{i_1 i}\delta_{j_1 j} - \delta^{i}{}_{j_1}\delta_{j_1 j}. \end{aligned} \tag{3.24}$$

One can then show by using the independence of $A^{i_1 i}{}_{j_1 j}$ on j_1 that

$$\sum_{j_1} A^{i_1 i}{}_{j_1 j} C^{j_1 I}{}_{k_1 J} = 0. \tag{3.25}$$

Finally, we note that

$$R_{13}\left(\frac{n-2}{2}\right)^{i_1 i}{}_{j_1 k} R_{23}(0)^{i_2 k}{}_{j_2 j} = 0 \qquad \text{if} \qquad \begin{array}{c} i_1 \neq i_2 \\ j_1 \neq j_2 \end{array}. \tag{3.26}$$

Putting together the relations in (3.22)–(3.26) and using the relation in (3.20), one can then show that

$$\Lambda(u)\overline{\Lambda}(-u) = \frac{1}{(n-2)^{2L}}(u^2-1)^L(4u^2-(n-2)^2)^L + u^L\Lambda_r(u), \tag{3.27}$$

where $\Lambda_r(u)$ is a remainder term that is yet to be determined. The relation in (3.27) is highly constraining. As was shown by Reshetikhin [39, 40], its solution is

$$\begin{aligned}\Lambda(u) = & \frac{1}{(n-2)^L}\Big[(u-1)^L(2u-n+2)^L H(u) + u^L(2u-n+4)^L F(u) + \\ & + u^L(2u-n+2)^L G(u)\Big]\,,\end{aligned} \tag{3.28}$$

where in order to satisfy (3.27) and crossing symmetry

$$\begin{aligned}H(u)\overline{H}(-u) &= 1\\ \overline{F}\left(-u+\frac{n-2}{2}\right) &= H(u)\\ \overline{G}\left(-u+\frac{n-2}{2}\right) &= G(u)\,.\end{aligned} \tag{3.29}$$

A solution for the first of these equations is

$$H(u) = \prod_{j=1}^{n_1}\frac{u-iu_{1,j}+1/2}{u-iu_{1,j}-1/2}\,, \tag{3.30}$$

where the number n_1 and the possible values $u_{1,m}$ will depend on the particular eigenstate. If $u_{1,m}$ is complex, then its conjugate must also be contained in the product.

The function $G(u)$ will be written as a sum

$$G(u) = \sum_{q=1}^{n-2} G_q(u)\,, \tag{3.31}$$

where

$$\overline{G}_{n-1-q}\left(-u+\frac{n-2}{2}\right) = G_q(u)\,. \tag{3.32}$$

Let us assume that $n=2k$. Then the various $G_q(u)$ are given by

$$\begin{aligned}G_q(u) &= \prod_{j=1}^{n_q}\frac{u-iu_{q,j}-q/2-1}{u-iu_{q,j}-q/2}\prod_{j=1}^{n_{q+1}}\frac{u-iu_{q+1,j}-q/2+1/2}{u-iu_{q+1,j}-q/2-1/2} \qquad 1\le q<k-2\\ G_{k-2}(u) &= \prod_{j=1}^{n_{k-2}}\frac{u-iu_{k-2,j}-k/2}{u-iu_{q,j}-k/2+1}\prod_{j=1}^{n_{k-1}}\frac{u-iu_{k-1,j}-k/2+3/2}{u-iu_{k-1,j}-k/2+1/2}\prod_{j=1}^{n_k}\frac{u-iu_{k,j}-k/2+3/2}{u-iu_{k,j}-k/2+1/2}\\ G_{k-1}(u) &= \prod_{j=1}^{n_{k-1}}\frac{u-iu_{k-1,j}-k/2+3/2}{u-iu_{k-1,j}-k/2+1/2}\prod_{j=1}^{n_k}\frac{u-iu_{k,j}-k/2+1/2}{u-iu_{k,j}-k/2+1/2}\,.\end{aligned} \tag{3.33}$$

However, the eigenvalues must be a polynomial in u, but given the structure of the above functions, it appears that $\Lambda(u)$ will have poles at $u=iu_{q,m}$ for all of the various values of q and m. Hence, there has to be intricate relations between the different values

$u_{q,m}$ in order that the poles cancel. The relations were derived in [39, 40] and are given by

$$\begin{aligned}
\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^L &= \prod_{j\neq i}^{n_1}\frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i}\prod_{j}^{n_2}\frac{u_{1,i}-u_{2,j}-i/2}{u_{1,i}-u_{2,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_q}\frac{u_{q,i}-u_{q,j}+i}{u_{q,i}-u_{q,j}-i}\prod_{j}^{n_{q-1}}\frac{u_{q,i}-u_{q-1,j}-i/2}{u_{q,i}-u_{q-1,j}+i/2}\prod_{j}^{n_{q+1}}\frac{u_{q,i}-u_{q+1,j}-i/2}{u_{q,i}-u_{q-1,j}+i/2}\\
& 1<q<k-2\\
1 &= \prod_{j\neq i}^{n_{k-2}}\frac{u_{k-2,i}-u_{k-2,j}+i}{u_{k-2,i}-u_{k-2,j}-i}\prod_{j}^{n_{k-3}}\frac{u_{k-2,i}-u_{k-3,j}-i/2}{u_{k-2,i}-u_{k-3,j}+i/2}\times\\
&\quad\times\prod_{j}^{n_{k-1}}\frac{u_{k-2,i}-u_{k-1,j}-i/2}{u_{k-2,i}-u_{k-1,j}+i/2}\prod_{j}^{n_k}\frac{u_{k-2,i}-u_{k,j}-i/2}{u_{k-2,i}-u_{k,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_{k-1}}\frac{u_{k-1,i}-u_{k-1,j}+i}{u_{k-1,i}-u_{k-1,j}-i}\prod_{j}^{n_{k-2}}\frac{u_{k-1,i}-u_{k-2,j}-i/2}{u_{k-1,i}-u_{k-2,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_k}\frac{u_{k,i}-u_{k,j}+i}{u_{k,i}-u_{k,j}-i}\prod_{j}^{n_{k-2}}\frac{u_{k,i}-u_{k-2,j}-i/2}{u_{k,i}-u_{k-2,j}+i/2}\,.
\end{aligned}\tag{3.34}$$

These are the analogs of the Bethe equations for the Heisenberg spin chain [37], and the solutions are often called the Bethe roots. It was subsequently shown, that these series of equations can be generalized to arbitrary groups in different representations [59]. The generalized equations are given by

$$\left(\frac{u_{q,i}+i\vec{\alpha}_q\cdot\vec{w}/2}{u_{q,i}-i\vec{\alpha}_q\cdot\vec{w}/2}\right)^L = \prod_{j\neq i}^{n_q}\frac{u_{q,i}-u_{q,j}+i\vec{\alpha}_q\cdot\vec{\alpha}_q/2}{u_{q,i}-u_{q,j}-i\vec{\alpha}_q\cdot\vec{\alpha}_q/2}\prod_{q'\neq q}\prod_{j}^{n_{q'}}\frac{u_{q,i}-u_{q',j}+i\vec{\alpha}_q\cdot\vec{\alpha}_{q'}/2}{u_{q,i}-u_{q',j}+i\vec{\alpha}_q\cdot\vec{\alpha}_{q'}/2}\,.\tag{3.35}$$

The different parameters $u_{q,i}$ are associated with the simple roots of the Lie group $\vec{\alpha}_q$, and the factor on the left hand side of the equations depend on the maximum weight of the representation, $\vec{w}$. In the case of SO$(2k)$ in the vector representation, we see that (3.34) has the form in (3.35). Finally, for the particular case of SO(6) where $k-2=1$, the first equation in (3.34) is modified to[3]

$$\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^L = \prod_{j\neq i}^{n_1}\frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i}\prod_{j}^{n_2}\frac{u_{1,i}-u_{2,j}-i/2}{u_{1,i}-u_{2,j}+i/2}\prod_{j}^{n_3}\frac{u_{1,i}-u_{3,j}-i/2}{u_{1,i}-u_{3,j}+i/2}.\tag{3.36}$$

The other two equations read

$$\begin{aligned}
1 &= \prod_{j\neq i}^{n_2}\frac{u_{2,i}-u_{2,j}+i}{u_{2,i}-u_{2,j}-i}\prod_{j}^{n_1}\frac{u_{2,i}-u_{1,j}-i/2}{u_{2,i}-u_{1,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_3}\frac{u_{3,i}-u_{3,j}+i}{u_{3,i}-u_{3,j}-i}\prod_{j}^{n_1}\frac{u_{3,i}-u_{1,j}-i/2}{u_{3,i}-u_{1,j}+i/2}\,.
\end{aligned}\tag{3.37}$$

[3] The simple roots of SO(6) are $\vec{\alpha}_1=(1,-1,0)$, $\vec{\alpha}_2=(0,1,-1)$, $\vec{\alpha}_3=(0,1,1)$, and the weight of the vector representation is $\vec{w}=(1,0,0)$.

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Now from (3.28) and (3.30) we can find the eigenvalues of the shift operator and the hamiltonian. The eigenvalues of the shift operator are

$$\Lambda(0) = H(0) = \prod_{i=1}^{n_1} \frac{u_{1,i} + i/2}{u_{1,i} - i/2}\,. \tag{3.38}$$

Hence the momenta of the eigenstates is

$$P = -i\log(\Lambda(0)) = -i\sum_i^{n_1} \log\frac{u_{1,i} + i/2}{u_{1,i} - i/2} = \sum_i^{n_1} p(u_{1,i})\,. \tag{3.39}$$

The corresponding energies are found from the eigenvalues of t_1, Λ_1, which are

$$\Lambda_1 = \frac{d}{du}H(u)\Big|_{u=0} - L\frac{n}{n-2}H(0)\,. \tag{3.40}$$

Using (3.16) and (3.17), we see that the energy eigenvalues are

$$E = \frac{n-2}{2H(0)}\frac{d}{du}H(u)\Big|_{u=0} = \frac{n-2}{2}\sum_i^{n_1}\epsilon(u_{1,i})\,, \tag{3.41}$$

where

$$\epsilon(u) = -\frac{d}{du}p(u) = 4\sin^2\left(\frac{p(u)}{2}\right) = \frac{1}{u^2+1/4}\,. \tag{3.42}$$

Hence the parameters $u_{1,i}$ are rapidity parameters for particle like excitations of the ground state.

Thus, specializing to SO(6) and using (2.4), (3.17) and (3.41), we find that the corresponding anomalous dimension is

$$\gamma = \frac{\lambda}{8\pi^2}\sum_{i=1}^{n_1}\epsilon(u_{1,i})\,. \tag{3.43}$$

4. Applying the Bethe ansatz

In this section we apply the results of the previous sections to many different scenarios. Sometimes we will reduce our space of operators to those involving just Z and W scalar fields.[4] In this case our problem is basically reduced to a Heisenberg spin chain. Including other fields complicates the problem somewhat, but we are still able to make many statements about the eigenvalues.

As we saw in the previous section, the SO(6) chain has three types of excitations, with each type associated with one of the simple roots of the SO(6) Dynkin diagram. Those associated with α_1 are on a somewhat different footing than those associated with α_2 and α_3, since only the α_1 excitations carry momentum and energy. However, the other two types of excitations can indirectly affect the energy of the state by modifying the u_1 rapidities.

[4]It is useful to combine six real fields into three complex scalars: $Z = \Phi_1 + i\Phi_2$, $W = \Phi_3 + i\Phi_4$, $Y = \Phi_5 + i\Phi_6$, which can be regarded as lowest components of three chiral superfields in the $\mathcal{N} = 1$ formalism.

If we were to limit ourselves to only u_1 excitations, then we see that the Bethe ansatz equations in (3.36) reduce to that of the ordinary Heisenberg spin chain. For this case, the different lattice sites can have one of two values (spin up or down). The Heisenberg spin chain has no trace term either, so the corresponding situation for the operator chains is to have two types of fields where the trace term does not contribute. So for example, we could have chains made up of Z and W terms only. If we call the ground state $\mathrm{tr}\, Z^J$, then the particle excitations with rapidities $u_{1,i}$ create W operators in the chain. Another way to see this is that the Z field is the highest weight in the vector representation of SO(6), which we write as $\vec{\mu}_1 = (1, 0, 0)$. Subtracting an $\vec{\alpha}_1 = (1, -1, 0)$ root then gives $(0, 1, 0)$ which corresponds to the W field.

Now suppose that we were to try and create u_2 and u_3 excitations without any u_1 excitations. It is not too hard to see from the Bethe equations that this is not possible. This is clear from the perspective of the group representations as well, since $\vec{\mu}_1 - \vec{\alpha}_2$ and $\vec{\mu}_1 - \vec{\alpha}_3$ are not SO(6) weights. However $\vec{\mu}_1 - \vec{\alpha}_1 - \vec{\alpha}_2$ and $\vec{\mu}_1 - \vec{\alpha}_1 - \vec{\alpha}_2$ are weights, so given some u_1 excitations, it is possible to have u_2 and u_3 excitations.

We should also note that our SO(6) lattice chain appears in a trace, which means that the corresponding wave functions are invariant under translation. Hence the total momentum is zero.So in all considerations we require the trace condition for the $u_{1,i}$

$$\prod_{i=1}^{n_1} \frac{u_{1,i} + i/2}{u_{1,i} - i/2} = 1\,. \tag{4.1}$$

4.1 Two impurities

We first consider the case of two impurities, that is two u_1 excitations, which we label as $u_{1,1}$ and $u_{1,2}$. We need at least two impurities if we want to have excitations with non-zero momentum, but with zero total momentum to satisfy the trace condition. With two impurities the bare dimension exceeds the R charge by two units: $L = J + 2$. From (3.39) we have that

$$\frac{u_{1,1} + i/2}{u_{1,1} - i/2} \frac{u_{1,2} + i/2}{u_{1,2} - i/2} = 1\,. \tag{4.2}$$

Recalling that $u_{1,2} = u_{1,1}^*$ unless they are both real, we see that the only solutions have $u_{1,2} = -u_{1,1}$ with both values real. Now using (3.36), we find

$$\left(\frac{u_{1,1} + i/2}{u_{1,1} - i/2}\right)^L = \frac{2u_{1,1} + i}{2u_{1,1} - i} \tag{4.3}$$

and so we find that

$$p(u_{1,1}) = \frac{2\pi n}{L - 1} = \frac{2\pi n}{J + 1} \tag{4.4}$$

and from (3.42)

$$\epsilon(u_{1,1}) = \frac{1}{u_{1,1}^2 + 1/4} = 4 \sin^2 \frac{\pi n}{J + 1}\,. \tag{4.5}$$

Therefore, using (3.42) and (3.43) the anomalous dimension for this configuration is

$$\gamma_n^S = \frac{\lambda}{16\pi^2}\frac{6-2}{2} \times 2\epsilon(u_{1,1}) = \frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{J+1}\,, \tag{4.6}$$

which agrees with the result in (2.23). With no u_2 or u_3 excitations, the impurities are both W's and so their representation is symmetric traceless.

On top of the u_1 impurities, we can also add up to one each of the u_2 and u_3 impurities in a nontrivial way. Putting in a u_2 impurity, we see that (4.2) is unchanged, so $u_{1,1} = -u_{1,2}$. Using (3.37), we also have that

$$1 = \frac{u_2 - u_{1,1} - i/2}{u_2 - u_{1,1} + i/2}\frac{u_2 + u_{1,1} - i/2}{u_2 + u_{1,1} + i/2}\,. \tag{4.7}$$

The only solutions to this are $u_2 = \infty$ and $u_2 = 0$. The first case is the trivial solution in that it gives us the same anomalous dimension as before. This corresponds to having a W and a Y in the symmetric representation. Taking the second solution and plugging it into (3.36), we find that

$$p(u_{1,1}) = \frac{2\pi n}{L} = \frac{2\pi n}{J+2}\,, \tag{4.8}$$

and the anomalous dimension is

$$\gamma_n^A = \frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{J+2}\,, \tag{4.9}$$

the result previously given in (2.24). This then is the antisymmetric combination of W and Y. This is part of the self-dual representation of the SO(4) subgroup.

If we now also add a u_3 impurity, then u_3 has an equation identical to that for u_2 in (4.7). If there is no u_2 impurity, then the anomalous dimension is the same as in (4.9). This is part of the anti-selfdual representation of SO(4). With both types of impurities, the nontrivial solutions then have $u_2 = u_3 = 0$ and so (3.36) gives

$$p(u_{1,1}) = \frac{2\pi n}{L+1} = \frac{2\pi n}{J+3}\,, \tag{4.10}$$

and the anomalous dimension is

$$\gamma_n = \frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{J+3}\,, \tag{4.11}$$

the result previously given in (2.28). Notice that $-\vec{\alpha}_2$ takes W to Y and $-\vec{\alpha}_3$ takes the W to $\overline{Y}$. But we also have that $-\vec{\alpha}_2 - \vec{\alpha}_3$ takes W to $\overline{W}$ and that $-2\vec{\alpha}_1 - \vec{\alpha}_2 - \vec{\alpha}_3$ takes a Z to $\overline{Z}$. Hence this last result corresponds to the SO(4) invariant of the two impurities.

4.2 More than two impurities

In this section we consider the addition of many impurities and compute their anomalous dimensions, up to first order in $1/J$. For the most part we will limit our discussion to having only u_1 excitations.Hence, these will only be a subset of possible SO(6) representations, namely, the real representations with $2L$ boxes in the SU(4) Young Tableaux. At the end of the section we will discuss the addition of a single u_2 or u_3 impurity.

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Once we have more than two impurities, it is now possible to have complex u_1 rapidities. In fact, this possibility is basically forced on us when we want to find BMN states where a particular oscillator appears more than once. In the BMN limit, the momenta of the excitations should be small, and so the phases in the Bethe equations are small. But if two excitations have identical momenta, then the combination $\frac{u_{1,1}-u_{1,2}+i}{u_{1,1}-u_{1,2}-i}$ which appears in the righthand side of the Bethe equations will have a large phase.

The resolution of this problem is that $u_{1,1}$ and $u_{1,2}$ get imaginary pieces such that $u^*_{1,2} = u_{1,1}$. This way we can get a small phase so long as $|\mathrm{Im} u_{1,1}| \gg 1$. The individual momenta of the excitations are complex, but the combined momentum

$$p(u_{1,1}, u_{1,2}) = p(u_{1,1}) + p(u_{1,2}) = -i \log \frac{u_{1,1} + i/2}{u_{1,1} - i/2} - i \log \frac{u^*_{1,1} + i/2}{u^*_{1,1} - i/2} \tag{4.12}$$

is real. Note further that the combined energy from these two excitations is

$$\epsilon(u_{1,1}) + \epsilon(u_{1,2}) = 4 \sin^2 \left(\frac{p(u_{1,1})}{2} \right) + 4 \sin^2 \left(\frac{p(u_{1,2})}{2} \right) \leq 8 \sin^2 \left(\frac{p(u_{1,1}, u_{1,2})}{4} \right), \tag{4.13}$$

where there is an equality only if the individual momenta are real. Hence, this configuration corresponds to a bound state of two particles,[5] since the combined energy is less than twice the energy of a single particle with momentum $p(u_{1,1}, u_{1,2})/2$. This can be generalized to many particles as well, where the individual momenta are complex, but their sum is real. So a BMN state with M oscillators at the same level would correspond to a bound state of M particles.

Unfortunately, it does not seem possible to find exact generic solutions to the Bethe equations for more than two excitations. However, it is possible to at least find $1/J$ corrections in the BMN limit. If we have particles with small momenta, then the values of $u_{1,i}$ are large. From the Bethe equations, we see to leading order that these are

$$u_{1,n} \approx \frac{L}{2\pi k_n}, \tag{4.14}$$

where k_n is an integer. Allowing for bound states, let us group the various excitations as $\mu_i^{(n)}$, where

$$\mu_i^{(n)} = \frac{1}{2\pi k_n} \left(L + i L^{1/2} \nu_i^{(n)} + \delta_i^{(n)} \right) + \mathrm{O}\left(L^{-1/2} \right). \tag{4.15}$$

We assume that $k_n \neq k_m$ if $n \neq m$ and the index i sums over the M_n particles making up the bound state at k_n. We can now expand the Bethe equations in (3.36), (3.37) in powers of $1/\sqrt{L}$. Solving for the zeroth order term in the expansion gives integer k_n.

Next solving for the $L^{-1/2}$ term in the expansion gives the equation

$$\nu_i^{(n)} = \sum_{j \neq i} \frac{2}{\nu_i^{(n)} - \nu_j^{(n)}}. \tag{4.16}$$

[5] If the momentum were of order 1, then the separation between $u_{1,1}$ and $u_{1,2}$ would be close to i. In the literature, these bound states are called "strings", but we will stick to calling them bound states for obvious reasons.

It turns out that we don't need to explicitly know the $\nu_i^{(n)}$ when computing the anomalous dimension to this order, but let us consider solutions of (4.16) for a few different values of M_n anyway. If $M_n = 2$, then we have that $\nu_1^{(n)} = -\nu_2^{(n)} = 1$. If $M_n = 3$, then $\nu_1^{(n)} = -\nu_2^{(n)} = \sqrt{3}$ and $\nu_3^{(n)} = 0$. Finally, let us consider the case where $M_n \gg 1$, then we can describe the distribution of $\nu^{(n)}$'s by a continuous density and may approximate the sum by an integral

$$\nu = P \int_{-a}^{a} d\nu' \, \frac{2\,\rho^{(n)}(\nu')}{\nu - \nu'} \tag{4.17}$$

which is Wigner's equation for the eigenvalues of a large N hermitean matrix model with a gaussian potential. Standard techniques give

$$\rho^{(n)}(\nu) = \frac{1}{2\pi}\sqrt{a^2 - \nu^2} \qquad\qquad a = 2\sqrt{M_n}\,. \tag{4.18}$$

Notice that if $M_n \sim L$, then the maximum value of $\nu_i^{(n)} \sim \sqrt{L}$ and so the ansatz in (4.15) breaks down.

Next solving for the L^{-1} term in the expansion of the Bethe equations leads to the equation

$$\delta_i^{(n)} + \left(\nu_i^{(n)}\right)^2 + 2\sum_{j\neq i} \frac{\delta_i^{(n)} - \delta_j^{(n)}}{\left(\nu_i^{(n)} - \nu_j^{(n)}\right)^2} + 2\sum_{m\neq n} M_m \frac{k_m}{k_m - k_n} = 0\,. \tag{4.19}$$

To solve this equation, we make the ansatz that

$$\delta_i^{(n)} = c_n (\nu_i^{(n)})^2 + b_n\,. \tag{4.20}$$

Substituting this back into (4.19) and making use of (4.16) we find that

$$\begin{aligned} c_n &= -\frac{1}{3} \\ b_n &= = -\frac{2}{3}(M_n - 1) - 2\sum_{m\neq n} \frac{M_m k_m}{k_m - k_n}\,, \end{aligned} \tag{4.21}$$

and so

$$\delta_i^{(n)} = -\frac{1}{3}\left(\nu_i^{(n)}\right)^2 - \frac{2}{3}(M_n - 1) - 2\sum_{m\neq n} \frac{M_m k_m}{k_m - k_n}\,. \tag{4.22}$$

Let us now place these results into the energies in (3.42) and (3.41). Up to and including corrections of order $1/L$, we can approximate these as

$$\epsilon_i^{(n)} = \frac{1}{(u_i^{(n)})^2}\,. \tag{4.23}$$

Hence, the energy coming from a single bound state is

$$\begin{aligned} \epsilon^{(n)} &= \sum_{i=1}^{M_n} \epsilon_i^{(n)} = \left(\frac{2\pi k_n}{L}\right)^2 \sum_i \left[1 - \frac{2}{L}\delta_i^{(n)} - \frac{3}{L}(\nu_i^{(n)})^2\right] \\ &= \left(\frac{2\pi k_n}{L}\right)^2 \frac{1}{L}\left[L M_n + \frac{4}{3}M_n(M_n - 1) + 4\sum_m \frac{M_n M_m k_m}{k_m - k_n} - \frac{7}{3}\sum_i (\nu_i^{(n)})^2\right]. \end{aligned} \tag{4.24}$$

Using (4.16), we have that

$$\sum_i (\nu_i^{(n)})^2 = M_n(M_n - 1)\,. \tag{4.25}$$

Putting this back in (4.24) we find

$$\epsilon^{(n)} = \left(\frac{2\pi k_n}{L}\right)^2 \frac{M_n}{L}\left[L - (M_n - 1) + 4\sum_{m\neq n}\frac{M_m k_m}{k_m - k_n}\right]. \tag{4.26}$$

The negative term inside (4.26) is basically the contribution of the binding energy, where the binding energy is present if $M_n > 1$. The last term in (4.26) comes from interactions among the different bound states.

The anomalous dimension is then found by adding up the $\epsilon^{(n)}$, giving

$$\gamma = \frac{\lambda}{2L^3}\sum_n M_n k_n^2\left[L - (M_n - 1) + 4\sum_{m\neq n}\frac{M_m k_m}{k_m - k_n}\right] + \mathrm{O}(L^{-4})\,. \tag{4.27}$$

The trace condition in (4.1) requires that

$$\sum_n M_n k_n = 0\,. \tag{4.28}$$

We can then use this to reduce (4.27) to

$$\gamma = \frac{\lambda}{2L^3}\sum_n M_n k_n^2 (L + M_n + 1) + \mathrm{O}(L^{-4})\,. \tag{4.29}$$

Since we have added n_1 impurities, $L = J + n_1$. Writing γ in terms of J we find

$$\gamma = \frac{\lambda}{2J^3}\sum_n M_n k_n^2 (J - 2n_1 + M_n + 1) + \mathrm{O}(J^{-4})\,. \tag{4.30}$$

Let us now add a single u_2 and/or a single u_3 impurity to the mix. We have not yet found an explicit formula analagous to (4.27) for a generic number of these impurities. With only one each of these impurities, the Bethe equations lead to

$$1 = \prod_{i=1}^{n_1}\frac{u_2 - u_{1,i} - i/2}{u_2 - u_{1,i} + i/2}\,, \tag{4.31}$$

and an identical equation for u_3. Hence, u_2 and u_3 can be determined in terms of $u_{1,i}$ by solving an order n_1 polynomial equation. This can be solved if n_1 is small, but in general the equation appears complicated. However, it is easy to see using the trace condition in (4.1) that $u_2 = 0$ and $u_3 = 0$ are always solutions to the equations.

If we only have one of the impurities, with its value set to 0, then we see that the Bethe equation in (3.36) is identical to the equation with no impurities, except that L is replaced with $L + 1$. So if the exact solution could be found for the case with only u_1 impurities, then the solution would be known for this case as well. Likewise, if we have both a $u_2 = 0$ and a $u_3 = 0$ impurity, then we should replace L by $L + 2$.

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5. Large excitations

Ultimately, we would like to solve string theory for the full $AdS_5 \times S_5$ and not just for the plane wave limit. Then one could compare the anomalous dimensions of all gauge invariant operators. Likewise, one would need to actually compute the dimensions of these operators in the field theory. So far, we have been restricting ourselves to large R-charge, where we are limited to finding $1/J$ corrections to BMN operators. We can think of this as the dilute gas limit [4].

Remarkably, the Bethe ansatz equations can be used to ascertain information about operators outside the BMN regime of validity. For example, one might ask what is the largest possible anomalous dimension for an operator (made up of scalars only) with engineering dimension L. This should be an SO(6) singlet. It turns out that this is solvable in the large L limit [45], with a solution similar to that of the ground state of the Heisenberg anti-ferromagnet. Of course the ground state of the Heisenberg anti-ferromagnet also corresponds to a particular SO(6) representation.

We now review the solution in [45] and then use the result to find the anomalous dimension. To find the solution, we assume a large number of excitations of all impurity types. To maximize the energy, we should take the maximal number of impurities such that the solutions to the Bethe equations are all real. If we take the log of (3.36) and (3.37), we find the equations

$$\begin{aligned} L\vartheta(2u_{1,j}) &= j\pi + \sum_{j\neq i}\vartheta(u_{1,i}-u_{1,j}) - \sum_{j}\vartheta(2(u_{1,i}-u_{2,j})) - \sum_{j}\vartheta(2(u_{1,i}-u_{3,j})) \\ 0 &= j\pi + \sum_{j\neq i}\vartheta(u_{2,i}-u_{2,j}) - \sum_{j}\vartheta(2(u_{2,i}-u_{1,j})) \\ 0 &= j\pi + \sum_{j\neq i}\vartheta(u_{3,i}-u_{3,j}) - \sum_{j}\vartheta(2(u_{3,i}-u_{1,j}))\,, \end{aligned} \tag{5.1}$$

where

$$\vartheta(u) = \arctan(u)\,. \tag{5.2}$$

Since the wave functions would be zero if $u_{1,i} = u_{1,j}$ with $i \neq j$, the various roots are pushed onto different branches of the arctangent.

If L is very large, then we can replace j/L by a continuous variable x and the Bethe roots by $u_1(x)$, $u_2(x)$ and $u_3(x)$. By symmetry, we expect the distribution of $u_{2,i}$ and $u_{3,i}$ to be identical, hence we set $u_2(x) = u_3(x)$. The equations in (5.1) now become

$$\begin{aligned} \vartheta(2u_1(x)) &= \pi x + \int dy\vartheta(u_1(x)-u_1(y)) - 2\int dy\vartheta(2(u_1(x)-u_2(y)) \\ 0 &= \pi x + \int dy\vartheta(u_2(x)-u_2(y)) - \int dy\vartheta(2(u_2(x)-u_1(y))\,. \end{aligned} \tag{5.3}$$

We now take derivatives with respect to $u_1(x)$ and $u_2(x)$, which gives us

$$\begin{aligned} \frac{2}{4u^2+1} &= \pi\rho_1(u) + \int_{-\infty}^{\infty} du'\frac{\rho_1(u')}{(u-u')^2+1} - 4\int_{-\infty}^{\infty} du'\frac{\rho_2(u')}{4(u-u')^2+1} \\ 0 &= \pi\rho_2(u) + \int_{-\infty}^{\infty} du'\frac{\rho_2(u')}{(u-u')^2+1} - 2\int_{-\infty}^{\infty} du'\frac{\rho_1(u')}{4(u-u')^2+1}\,, \end{aligned} \tag{5.4}$$

where the $\rho_1(u)$ and $\rho_2(u)$ are the root densities

$$\rho_1(u) = \frac{dx}{du_1(x)}\Big|_{u_1(x)=u} \qquad \rho_2(u) = \frac{dx}{du_2(x)}\Big|_{u_2(x)=u}. \tag{5.5}$$

To verify that this root configuration is the SO(6) singlet, we can take the large u limit of (5.4) and assume that $\rho_1(u)$ and $\rho_2(u)$ fall off faster than u^{-2} as $u \to \infty$. This shows that

$$\int_{-\infty}^{\infty} du' \rho_1(u') = 1 \qquad \int_{-\infty}^{\infty} du' \rho_1(u') = \frac{1}{2}, \tag{5.6}$$

which means that there are L u_1 impurities and $L/2$ u_2 and u_3 impurities, precisely what is needed to take the large J state to the singlet.

We can solve for $\rho_1(u)$ and $\rho_2(u)$ in (5.4) by Fourier transforming. Defining

$$\widetilde{\rho}_1(k) = \int du \exp(iku)\rho_1(u) \qquad \widetilde{\rho}_2(k) = \int du \exp(iku)\rho_2(u), \tag{5.7}$$

it is straightforward to show that the solutions of (5.4) are

$$\widetilde{\rho}_1(k) = \frac{\cosh(k/2)}{\cosh(k)} \qquad \widetilde{\rho}_2(k) = \frac{1}{2\cosh(k)}, \tag{5.8}$$

Transforming back gives us

$$\rho_1(u) = \frac{\cosh(u\pi/2)}{\sqrt{2}\cosh(u\pi)} \qquad \rho_2(u) = \frac{1}{4\cosh(u\pi/2)}. \tag{5.9}$$

The anomalous dimension can now be computed and is

$$\gamma = \frac{\lambda}{8\pi^2}E = \frac{\lambda}{8\pi^2}L\int_{-\infty}^{\infty} du \frac{\rho_1(u)}{u^2+1/4} = \frac{\lambda}{8\pi^2}L\left(\frac{\pi}{2}+\ln 2\right). \tag{5.10}$$

Not surprisingly, the anomalous dimension is extensive: it depends linearly on L. However, recall that in the BMN limit, we saw that two impurities with the same real momentum had to have their roots split off from the real line. Hence if all the roots are real, each u_1 impurity has to correspond to a string oscillator with a different level number. Since there are L such impurities and since they are equally distributed between left and right oscillators, we find that the total level ℓ_{tot} is

$$\ell_{\mathrm{tot}} = \sum_{\ell=1}^{L/2} \ell \approx \frac{L^2}{8}. \tag{5.11}$$

Therefore, we see that the *full* dimension of the operator has the behavior

$$\Delta = L + \gamma = \sqrt{\ell_{\mathrm{tot}}}\left(2\sqrt{2} + \frac{\lambda}{2\sqrt{2}\pi^2}\left(\frac{\pi}{2}+\ln 2\right)\right) + \mathrm{O}(\lambda^2), \tag{5.12}$$

the same square root dependence on the level that is generic for small α' in string theory [2]. Of course, small α' corresponds to strong coupling where the dimension of the operator had a $(\lambda)^{1/4}$ dependence. In any event, (5.12) suggests that the square root dependence of the

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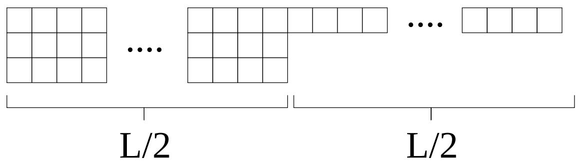

Figure 2: Young tableau corresponding to the antiferromagnet configuration.

level is generic, even at weak coupling. Note that corrections coming from higher orders in perturbation theory should also give contributions to the dimension which are linear in L, since the large N expansion essentially localizes the interactions to nearby neighbors.

Although the level square root dependence appears to be generic, the actual λ dependence depends on the operator under consideration. For example, let us consider the operator whose SU(4) Young tableau is shown in figure 2. The corresponding Bethe state has $L/2$ u_1 excitatations and no u_2 and u_3 excitations. We then have the first equation in (5.4) but with $\rho_2(u) = 0$. This is same equation found for the anti-ferromagnetic Heisenberg spin chain. Its solution is well known (e.g. see [56]). The anomalous dimension is

$$\gamma = \frac{\lambda}{8\pi^2}E = \frac{\lambda}{8\pi^2}L\int_{-\infty}^{\infty} du\frac{\rho_1(u)}{u^2+1/4} = \frac{\lambda}{4\pi^2}L\ln 2\,. \tag{5.13}$$

The anomalous dimension is smaller than in (5.10), but so is the level, since there are only $L/2$ excitations. For this particular state, we see that the full dimension is

$$\Delta = L+\gamma = \sqrt{\ell_{\rm tot}}\left(4\sqrt{2}+\frac{\lambda\sqrt{2}}{\pi^2}\ln 2\right)+\mathrm{O}(\lambda^2)\,. \tag{5.14}$$

Thus, this has the level square root dependence, but the λ dependence is different than that in (5.12).

One can also consider "excitations" [45, 57] away from this SO(6) singlet by including "holes" in the integers appearing in (5.1). The inclusions of these holes modifies the equations in (5.4) to

$$\begin{aligned}
\frac{2}{4u^2+1} &= \pi\rho_1(u)+\pi\sum_{j=1}^{\widetilde{n}_1}\delta(u-\widetilde{u}_{1,j})+\int_{-\infty}^{\infty}du'\frac{\rho_1(u')}{(u-u')^2+1}- \\
&\quad -2\int_{-\infty}^{\infty}du'\frac{(\rho_2(u')+\rho_3(u'))}{4(u-u')^2+1} \\
0 &= \pi\rho_2(u)+\pi\sum_{j=1}^{\widetilde{n}_2}\delta(u-\widetilde{u}_{2,j})+\int_{-\infty}^{\infty}du'\frac{\rho_2(u')}{(u-u')^2+1}- \\
&\quad -2\int_{-\infty}^{\infty}du'\frac{\rho_1(u')}{4(u-u')^2+1} \\
0 &= \pi\rho_3(u)+\pi\sum_{j=1}^{\widetilde{n}_3}\delta(u-\widetilde{u}_{3,j})+\int_{-\infty}^{\infty}du'\frac{\rho_2(u')}{(u-u')^2+1}- \\
&\quad -2\int_{-\infty}^{\infty}du'\frac{\rho_1(u')}{4(u-u')^2+1}\,,
\end{aligned} \tag{5.15}$$

where $\widetilde{n}_i$ refers to the number of holes of type i and $\widetilde{u}_{i,j}$ are the positions of the holes. Assuming that $\widetilde{n}_i \ll L$, the corrections from the δ-functions to the densities are additive, so we can consider them individually. It is convenient to write the densities as

$$\rho_1(u) = \rho_1^{(0)}(u) + \frac{1}{L}\sigma_1(u) \qquad \rho_2(u) = \rho_2^{(0)}(u) + \frac{1}{L}\sigma_2(u) \qquad \rho_3(u) = \rho_3^{(0)}(u) + \frac{1}{L}\sigma_3(u)\,, \tag{5.16}$$

where $\rho_i^{(0)}$ are the densities with no holes present.

For a hole of type 1 at position $\widetilde{u}_1$, we can write

$$\sigma_1(u) = \sigma_1^1(u - \widetilde{u}_1) \qquad \sigma_2(u) = \sigma_2^1(u - \widetilde{u}_1) \qquad \sigma_3(u) = \sigma_3^1(u - \widetilde{u}_1)\,. \tag{5.17}$$

The equations in (5.15) become

$$\begin{aligned} 0 &= \pi\sigma_1^1(u) + \pi\delta(u) + \int_{-\infty}^{\infty} du' \frac{\sigma_1^1(u')}{(u-u')^2+1} - 4\int_{-\infty}^{\infty} du' \frac{\sigma_2^1(u')}{4(u-u')^2+1} \\ 0 &= \pi\sigma_1^1(u) \int_{-\infty}^{\infty} du' \frac{\sigma_2^1(u')}{(u-u')^2+1} - 2\int_{-\infty}^{\infty} du' \frac{\sigma_1^1(u')}{4(u-u')^2+1}\,, \end{aligned} \tag{5.18}$$

where we have used the symmetry of the configuration to set $\sigma_2^1(u) = \sigma_3^1(u)$. The equations in (5.18) are easily solved, giving

$$\begin{aligned} \sigma_1^1(u) &= -\int \frac{dk}{2\pi} e^{-iku} \frac{e^{-|k|/2}\cosh(k/2)}{\cosh(k)} \\ \sigma_2^1(u) &= \sigma_3^1(u) = -\int \frac{dk}{2\pi} e^{-iku} \frac{e^{-|k|/2}}{2\cosh(k)}\,. \end{aligned} \tag{5.19}$$

The change in the energy is

$$\epsilon(\widetilde{u}_1) = \int_{-\infty}^{\infty} du \frac{\sigma_1(u-\widetilde{u}_1)}{u^2+1/4} = -2\pi\rho_1^{(0)}(\widetilde{u}_1)\,, \tag{5.20}$$

where $\rho_1^{(0)}(u)$ is the solution in (5.9).

To find the momentum of the hole, we can integrate $\epsilon(\widetilde{u}_1)$ with respect to $\widetilde{u}_1$, giving

$$p(\widetilde{u}_1) = \pi - 2\arctan\left(\sqrt{2}\sinh\frac{\widetilde{u}_1\pi}{2}\right). \tag{5.21}$$

Notice that $Lp(\widetilde{u}_1)/(2\pi)$ is the change in the level coming from the introduction of the hole, which can be easily deduced by looking at (5.1), (5.4) and (5.20). It is also possible to express ϵ in terms of p, where we find

$$\epsilon(p) = -\pi\sin\left(\frac{p}{2}\right)\sqrt{1+\sin^2\left(\frac{p}{2}\right)} \qquad 0 \le p < 2\pi\,, \tag{5.22}$$

and so the change in the anomalous dimension is

$$\Delta\gamma = -\frac{\lambda}{8\pi}\sin\left(\frac{p}{2}\right)\sqrt{1+\sin^2\left(\frac{p}{2}\right)}. \tag{5.23}$$

In order to understand the nature of these holes, notice that

$$\int du \sigma_1^1(u) = 1\,, \qquad \int du \sigma_2^1(u) = \int du \sigma_3^1(u) = \frac{1}{2}\,. \tag{5.24}$$

Hence, we need an even number of these types of holes.[6] We also see that the highest weight of each hole is $\vec{w} = \vec{\alpha}_1 + \frac{1}{2}(\vec{\alpha}_2 + \vec{\alpha}_3)$ which is the highest weight of the SO(6) vector representation. Hence these holes come with a vector index.

Next consider a type 2 hole. Proceeding as before, we find that

$$\begin{aligned}
\sigma_1^2(u) &= -\int \frac{dk}{2\pi} e^{-iku} \frac{e^{-|k|/2}}{1+e^{-2|k|}} \\
\sigma_2^2(u) &= -\int \frac{dk}{2\pi} e^{-iku} \frac{1+e^{-|k|}+e^{-2|k|}}{(1+e^{-|k|})(1+e^{-2|k|})} \\
\sigma_3^2(u) &= -\int \frac{dk}{2\pi} e^{-iku} \frac{e^{-|k|}}{(1+e^{-|k|})(1+e^{-2|k|})}\,.
\end{aligned} \tag{5.25}$$

The energy of this type of hole is

$$\epsilon(\widetilde{u}_2) = \int du \frac{\sigma_1^2(u-\widetilde{u}_2)}{u^2+1/4} = -2\pi\rho_2^{(0)}(\widetilde{u}_2)\,, \tag{5.26}$$

where $\rho_2^{(0)}(\widetilde{u}_2)$ is the density in (5.9). Integrating ϵ, we find that the momentum is

$$p(\widetilde{u}_2) = \pi - 2\arctan(e^{\pi\widetilde{u}/2})\,, \tag{5.27}$$

and so the energy of this type of hole in terms of p is

$$\epsilon(p) = -\frac{\pi}{2}\sin p \qquad 0 \le p \le \pi\,. \tag{5.28}$$

Hence these holes occupy only half of a Brillouin zone. We also have that

$$\int du \sigma_1^2(u) = \frac{1}{2}, \qquad \int du \sigma_2^2(u) = \frac{3}{4} \qquad \int du \sigma_3^2(u) = \frac{1}{4}\,, \tag{5.29}$$

thus the highest weight of each type 2 hole is $\vec{w} = \frac{1}{2}\vec{\alpha}_1 + \frac{3}{4}\vec{\alpha}_2 + \frac{1}{4}\vec{\alpha}_3$ which is the highest weight of one of the spinor representations.

The argument for type 3 holes is the same as for type 2. The highest weight of each type 3 hole is $\vec{w} = \frac{1}{2}\vec{\alpha}_1 + \frac{1}{4}\vec{\alpha}_2 + \frac{3}{4}\vec{\alpha}_3$, hence each of these type holes is in the other spinor representation. Since the two spinor representations are complex conjugates, we choose the energies of the type 3 holes to be

$$\epsilon(p) = +\frac{\pi}{2}\sin p \qquad \pi \le p \le 2\pi\,. \tag{5.30}$$

The trace condition forces the total momentum of the holes to be zero mod 2π. We can also see from (5.29) that every type 2 hole has to either come with three other type 2 holes, or a type 3 hole. The same is true for type 3 holes. These conditions tell us that we cannot have individual spinor excitations, since the chain itself is made up of SO(6) vectors. Instead the representations have to combine to form an adjoint rep, or another representation that is trivial under the SO(6) center.[7]

[6] We can have an odd number, but we need to add another lattice site.

[7] If L is odd, then the excitations combine to form a vector representation, or another representation which has the same action under the center of SO(6).

6. Conclusions

In this paper we constructed a mixing operator for anomalous dimensions and showed that it was related to the hamiltonian of an integrable SO(6) chain. We then used the Bethe ansatz to find the anomalous dimensions of many operators, including those that were outside the BMN limit. We also demonstrated that these non-BMN operators have anomalous dimensions that depend on the square root of the level, a result also found at strong coupling.

There are many other operators where it is hoped that the Bethe ansatz will allow one to compute anomalous dimensions. These include the operators that correspond to large wound strings oscillating on the S^5. A prediction was made for the anomalous dimensions based on a semiclassical analysis [60], and it would be nice to explicitly verify this.

It would also be nice if one could somehow relate the higher loop corrections to integrable hamiltonians. One possibility is that the higher loop corrections correspond to the higher hamiltonians in the heirarchy of the same spin chain. On one level, this seems reasonable. In the large N limit, one would expect the g loop corrections to the anomalous dimensions to involve mixing between $g+1$ nearest neighbors, which is precisely what is found in the g^{th} hamiltonian in the heirarchy. However, this idea does not appear to work. For example, the two-loop analysis as done in [11] shows that the two-loop anomalous dimension matrix should have operators of the form

$$\sum_{l}^{L}(P_{l,l+1}P_{l+1,l+2}+P_{l+1,l+2}P_{l,l+1})\,. \tag{6.1}$$

But the next hamiltonian in the hierarchy of a Heisenberg system has the form

$$\sum_{l}^{L}(iP_{l,l+1}P_{l+1,l+2}-iP_{l+1,l+2}P_{l,l+1})\,. \tag{6.2}$$

That this idea does not work is perhaps not too surprising. If the higher hamiltonians of the hierarchy were indeed related to anomalous dimensions at higher orders of perturbation theory, then the mixing matrix could have been diagonalized by a unitary transformation which is independent of the coupling — a rather exceptional property. In any event, one can now ask what role the higher hamiltonians play on the gauge theory side.

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